

## Research



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# Consensus and clustering in opinion formation on networks

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Ideas that challenge the status quo either evaporate or dominate. The study of opinion dynamics in the socio-physics literature treats space as uniform and considers individuals in an isolated community, using ordinary differential equation (ODE) models. We extend these ODE models to include multiple communities and their interactions. These extended ODE models can be thought of as being ODEs on directed graphs. We study in detail these models to determine conditions under which there will be consensus and pluralism within the system. Most of the consensus/pluralism analysis is done for the case of one and two cities. However, we numerically show for the case of a symmetric cycle graph that an elementary bifurcation analysis provides insight into the phenomena of clustering. Moreover, for the case of a cycle graph with a hub, we discuss how having a sufficient proportion of zealots in the hub leads to the entire network sharing the opinion of the zealots.

This article is part of the theme issue 'Stability of nonlinear waves and patterns and related topics'.

## 1. Introduction

Models of opinion formation in the socio-physics literature examine the dynamics of beliefs [1,2], language [3–5], culture [6], voting preference [7–9] and rumour propagation [10]. Specifically, these models examine how individuals within communities influence each other and reach (or drift from) consensus [11,12]. Key to understanding these dynamics is the structure which defines how individuals interact. Social and complex networks are used as models, e.g. the

- (a) naming game [3–5,13,14],
- (b) voting model [7–9],
- (c) Bass model of innovation diffusion [15] and
- (d) Bayesian models of ‘trust’ [16].

We point the reader to [17] for an in-depth summary of these models. Cellular automata [15] and agent-based modelling have been used [2,18] to study the dynamics of opinion formation. On the other hand, a mean-field model approach can be used if the goal is not to understand the dynamics of opinion formation in an individual, but instead how proportions of the total population change over time. The spread of ideas has been compared to epidemiological models, like the SIR (susceptible–infectious–recovered) model [10,19]. Socio-physical models draw upon models in statistical physics, like the Ising model [20–22]. Our work draws on the mean-field model derived in Marvel *et al.* [23]. We are not the first to draw from this model. Variations of this model appear in which parameters are added to control for the ‘charisma’ [8] of the agents, their ‘friendliness’ [24] and stubbornness [25]. An underlying assumption for this mean-field model is that all interactions occur on the same time scale.

We propose and analyse an extension of the mathematical mean-field model provided in [23]. In particular, the mean-field model proposed by Marvel *et al.* [23] is a three-state model of opinion formation. The three states are  $A$ ,  $B$  (not  $A$ ) and  $AB$  (undecided). We will henceforth call a group for which the three-state model holds a *city*. The interactions between cities will be modelled as an ordinary differential equation (ODE) system on a graph, where the dynamics on each node are those for the three-state model, and the dynamics between the nodes follows an interaction rule which is similar to that which leads to the three-state model. Our model resembles [26], in which laws of motion for the probabilities of each individual holding an opinion are derived (also see [27], where the modelling and analysis draw from ideas in low-temperature thermodynamics).

A primary motivation for our model is the question of spatial variation in opinion formation. In the USA, any map associated with a presidential election makes clear that, while there is often local (county, zip code) consensus on a candidate, there is huge spatial variation in voting preference. In particular, there are clusters of local communities which will agree on a candidate. Governing this clustering phenomena is the fact that opinions within a community are not solely governed by the interactions within that same community. Outside voices, e.g. the media, scattered friends and family, migrants and tourists, all play a role in the formation of opinions at a local level. Therefore, isolated communities can be thought to be connected by weighted, directed edges where each incoming edge weight is the probability of interacting with individuals from the source community. Our model allows for the possibility that a city like New York can exert more influence on Uncertain, Texas [28] than Uncertain does on New York.

In our analysis, we address the question of whether or not there is *pluralism*. As defined by the Oxford Dictionaries, pluralism is ‘a condition or system in which two or more states, groups, principles, sources of authority, etc., coexist.’ Pluralism is often discussed within the contexts of politics, religion, sociology and philosophy, among other arenas. In the context of this paper, pluralism will mean a steady state of the mean-field ODE where there is no majority opinion. Alternatively, when we say the system has consensus, that means that there is a majority opinion associated with the steady-state solution. In the discussion surrounding the mathematical

definition of pluralism in §2c, we briefly discuss how the results contained herein may change if we change the threshold for consensus.

The paper is organized as follows. In §2, we look at the three-state ODE model for one city, and analyse it in terms of consensus and pluralism. In this analysis, we say the city has consensus if more than 50% of the population agrees on an opinion; otherwise, there is pluralism. In §3, we extend the original ODE model to an ODE model on a directed graph. The underlying assumption here is that groups in one city interact with groups in another city. In §4, we analyse the case of two interacting cities, and study how the presence of zealots (those who hold an opinion, and will not change their mind), and the interaction between the cities, influence the types of consensus that are possible. In §5, we consider the dynamics associated with having many interacting cities. Herein, we combine a bifurcation analysis with our consensus/pluralism analysis to better understand the condition(s) which lead to clustering of opinion on a network in which there are no zealots. We also consider the case when one of the cities contains zealots. For the ease of presentation, we consider only a network on a cycle graph in the first case, and a cycle graph with a hub in the second case. However, the ideas naturally generalize to networks on other types of graphs. Finally, in §6 we briefly summarize our results and provide some questions for future research.

## 2. The one-city model

### (a) ODE model

Our general model considers interactions between multiple communities, which hereafter will be referred to as cities. The base model for one city will be that presented in [23]. The base model is driven by dyadic interactions between individuals who hold opinions  $A$ ,  $B$  (which is ‘not  $A$ ’) and the moderate, or undecided,  $AB$ . Additionally, there are zealots,  $P$  of believers of  $A$  and  $Q$  of believers of  $B$ , who do not change their opinions after interacting with others. The interactions are governed by the set of rules presented in table 1.

For the ODE model we set:

- (a)  $a$ , proportion of the population holding opinion  $A$  whose opinion can change;
- (b)  $b$ , proportion of the population holding opinion  $B$  whose opinion can change;
- (c)  $p$ , proportion of the population who are zealots in  $A$ ;
- (d)  $q$ , proportion of the population who are zealots in  $B$ ;
- (e)  $m$ , proportion of the population holding opinion  $AB$ .

All of these numbers are non-negative. The number of moderates,  $m$ , is related to the other populations via

$$m = m(a, b, p, q) = 1 - (a + p) - (b + q).$$

As all of the variables are proportions, the variables must satisfy the following bounds:

$$0 \leq a + p \leq 1, \quad 0 \leq b + q \leq 1 \quad \text{and} \quad 0 \leq m \leq 1. \quad (2.1)$$

Taking the limit of a large population and a small time step between interactions, the governing rate equation (after a possible time renormalization) is the ODE

$$\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \mathbf{M} \begin{pmatrix} a + p \\ b + q \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} m & -a \\ -b & m \end{pmatrix}. \quad (2.2)$$

Here overdot denotes differentiation with respect to time. For the ODE model (2.2), it is not difficult to check that if the initial conditions satisfy the physical bounds of (2.1), then the bounds will be satisfied for all  $t \geq 0$ .

**Table 1.** Interactions that change the membership in each of the subpopulations. The subpopulations of zealots,  $P$  and  $Q$ , are constant.

speaker	listener pre-interaction	listener post-interaction
$A, P$	$B$	$AB$
	$AB$	$A$
$B, Q$	$A$	$AB$
	$AB$	$B$

## (b) Fixed point analysis

To classify regions in parameter space for which there is consensus or pluralism, we must analyse the fixed points for the ODE model. If the fixed point is stable, then it is an attractor, and realizable for an appropriate set of initial conditions. If the fixed point is unstable, then its unstable manifold divides the phase space, and allows for multiple stable fixed points to be realized (depending upon the initial condition). In Marvel *et al.* [23], there was a fixed point analysis of (2.2) under the assumption that opinion  $B$  has no zealots,  $q = 0$ . This analysis was later extended by Wang *et al.* [29] for the case that both populations had zealots. For the sake of clarity we now recreate some of that analysis here.

The fixed points are found by solving a coupled system of nonlinear equations,

$$0 = \det(\mathbf{M}) = m^2 - ab, \quad 0 = \mathbf{M} \begin{pmatrix} a+p \\ b+q \end{pmatrix}. \quad (2.3)$$

We first note that no matter the number of zealots, the proportion of moderates is no more than a third of the total population:

**Proposition 2.1.** *At a fixed point,  $0 \leq m \leq \frac{1}{3}$ .*

*Proof.* Since  $m^2 = ab$ , by the arithmetic mean–geometric mean inequality,

$$m = \sqrt{ab} \leq \frac{1}{2}(a+b).$$

Upon using the inequality,  $a+b+m = 1-p-q \leq 1$ ,

$$m + \frac{1}{2}m \leq \frac{1}{2}(a+b+m) \leq \frac{1}{2} \quad \rightsquigarrow \quad \frac{3m}{2} \leq \frac{1}{2} \quad \rightsquigarrow \quad m \leq \frac{1}{3}. \quad \blacksquare$$

## (i) One population of zealots

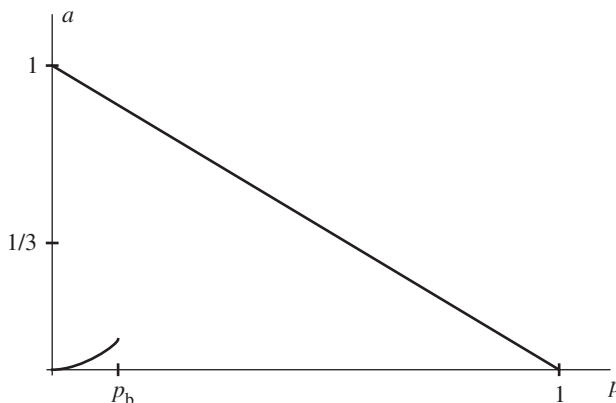
First assume that only opinion  $A$  has committed believers,  $q = 0$  (this situation was already covered in [23]). We will consider this scenario again in §4 when, in the two-city model, we analyse the situation where one city has zealots, but the other city does not. When  $p = 0$ , the fixed points are  $(1, 0)$ ,  $(0, 1)$ ,  $(\frac{1}{3}, \frac{1}{3})$ . The first two fixed points are stable nodes, and the last fixed point is a saddle point whose unstable manifold acts as a separatrix in the  $(a, b)$ -phase space. For  $p > 0$  the continuation of the first fixed point is  $(1-p, 0)$ , and it is a stable node for all values of  $p$ . We consider the fixed points that emanate from the other two.

**Proposition 2.2.** *When  $q = 0$ , an additional set of two fixed points for  $0 \leq p \leq p_b$ , where*

$$p_b = 1 - \frac{\sqrt{3}}{2} \sim 0.1340$$

*is given by  $(a, b) = (a_{\pm}, b_{\pm})$ , where*

$$a_{\pm} = \frac{1}{6} \left( 1 - 4p \pm \sqrt{1 - 8p + 4p^2} \right), \quad b_{\pm} = 1 - 2p - 2a_{\pm}.$$



**Figure 1.** A cartoon of the stability diagram for the fixed points of the system (2.2) when  $q = 0$ . The vertical axis is the proportion of non-zealots who hold opinion A. The curves represent the  $a$ -values associated with stable fixed points. Here,  $p_b \sim 0.1340$  represents the saddle-node bifurcation point between a stable node and a saddle point. There is always consensus. For  $p > p_b$ , there is consensus only for opinion A, while for  $p < p_b$  there can be consensus for either opinion.

The fixed point  $(a_-, b_-)$  is a stable node, whereas  $(a_+, b_+)$  is a saddle point (see figure 1 for a plot of the  $a$ -values of the stable fixed points).

*Proof.* Setting

$$x = \sqrt{a} \quad \text{and} \quad y = \sqrt{b},$$

the first equation in (2.3) becomes

$$m = xy \rightsquigarrow x^2 + xy + y^2 = 1 - p, \quad (2.4)$$

and the second and third equations collapse to

$$xy(x^2 + p - xy) = 0.$$

The solution  $y = 0$  leads to  $x^2 = 1 - p$ , which has already been discussed. Therefore, we are only interested in the other solution,

$$y = \frac{x^2 + p}{x} \rightsquigarrow b = \frac{(a + p)^2}{a}.$$

Plugging this solution into (2.4) yields the alternative expression presented in the proposition statement,

$$y^2 = 1 - 2p - 2x^2 \rightsquigarrow b = 1 - 2p - 2a.$$

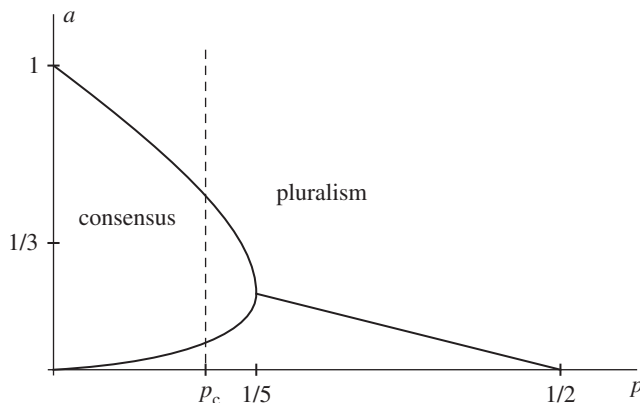
Plugging this expression for  $y$  back into (2.4) and simplifying leads to the quartic equation

$$3x^4 - (1 - 4p)x^2 + p^2 = 0 \rightsquigarrow x^2 = \frac{1}{6} \left( 1 - 4p \pm \sqrt{1 - 8p + 4p^2} \right).$$

As

$$1 - 8p + 4p^2 = 4(p - p_-)(p - p_+), \quad p_{\pm} = 1 \pm \frac{\sqrt{3}}{2},$$

we know this fixed point is valid only for  $p \leq p_- = p_b$ . The proof for the analytic expression of the fixed point is now complete. The stability argument is left as an exercise for the interested reader. ■



**Figure 2.** A cartoon of the stability diagram for the fixed points of the system (2.2) when  $p = q$ . The vertical axis is the proportion of non-zealots who hold opinion  $A$ . Here,  $p_c \sim 0.1910$  represents the transition point between consensus and pluralism (the pitchfork bifurcation of fixed points occurs at  $p = 0.2$ ). There is consensus for  $p < p_c$ , and pluralism otherwise.

## (ii) Two populations of zealots

We now suppose that each opinion has an equal number of zealots,  $p = q$  (the problem with  $p \neq q$  is considered in [29]). This is also what will be assumed when we consider a two-city model in §4. For  $p > 0$ , the continuation of the fixed point  $(\frac{1}{3}, \frac{1}{3})$  is

$$(a_0, b_0) = \left( \frac{1-2p}{3}, \frac{1-2p}{3} \right).$$

This follows from the fact that the fixed point equations (2.3) collapse to

$$m = a \rightsquigarrow 1 - 2a - 2p = a \rightsquigarrow a = \frac{1}{3}(1 - 2p).$$

Clearly, the fixed point is only physical for  $0 \leq p \leq \frac{1}{2}$ . The limit  $p = q = \frac{1}{2}$  corresponds to each opinion having only zealots. Regarding the stability of this fixed point, we have the following proposition.

**Proposition 2.3.** *The fixed point  $(a_0, b_0)$  is a saddle point for  $0 \leq p < \frac{1}{5}$ , and a stable node for  $\frac{1}{5} \leq p \leq \frac{1}{2}$ . There is a bifurcation of critical points at  $p = \frac{1}{5}$ .*

*Proof.* The linearization of the vector field at the fixed point is

$$A = \frac{1}{3} \begin{pmatrix} -(1+4p) & p-2 \\ p-2 & -(1+4p) \end{pmatrix},$$

which has the eigenvalues

$$\lambda_1 = -(1-p) \quad \text{and} \quad \lambda_2 = \frac{1}{3}(1-5p).$$

The stability result follows upon noting that  $\lambda_1 < 0$ , and  $\lambda_2$  changes sign at  $p = \frac{1}{5}$ . The existence of the bifurcation follows from  $\lambda_2$  changing sign at  $p = \frac{1}{5}$ . ■

The fixed point  $(a_0, b_0)$  exists for all values of  $p$ . Regarding the existence of the additional fixed points that emanate from the bifurcation at  $p = \frac{1}{5}$  (figure 2), we have the following proposition:

**Proposition 2.4.** *When  $p = q$ , an additional set of two fixed points for  $0 \leq p \leq \frac{1}{5}$  is given by  $(a, b) = (a_{\pm}, a_{\mp})$ , where*

$$a_{\pm} = \frac{1}{2} \left( 1 - 3p \pm \sqrt{(1-p)(1-5p)} \right).$$

*Each of these fixed points is a stable node.*

*Proof.* Setting

$$x = \sqrt{a} \quad \text{and} \quad y = \sqrt{b},$$

the first equation in (2.3) can be rewritten as

$$m = xy \rightsquigarrow x^2 + xy + y^2 = 1 - 2p, \quad (2.5)$$

and the second and third equations collapse to

$$y(x^2 + p) - x(y^2 + p) = 0 \rightsquigarrow (xy - p)(x - y) = 0.$$

The solution  $x = y$  was already covered in proposition 2.3, so we are only interested in the other solution,

$$y = \frac{p}{x} \rightsquigarrow b = \frac{p^2}{a}.$$

Alternatively, plugging this back into (2.5) gives

$$y^2 = 1 - 3p - x^2 \rightsquigarrow b = 1 - 3p - a.$$

Plugging the first expression for  $y$  back into (2.5) and simplifying leads to the quartic equation

$$x^4 - (1 - 3p)x^2 + p^2 = 0 \rightsquigarrow x^2 = \frac{1}{2} \left( 1 - 3p \pm \sqrt{(1 - p)(1 - 5p)} \right).$$

As  $a = x^2$ , upon using  $b = 1 - 3p - a$  we get the desired expression for the  $b$ -component of the fixed point. The proof for the analytic expression of the fixed point is now complete. The stability argument is left as an exercise for the interested reader. ■

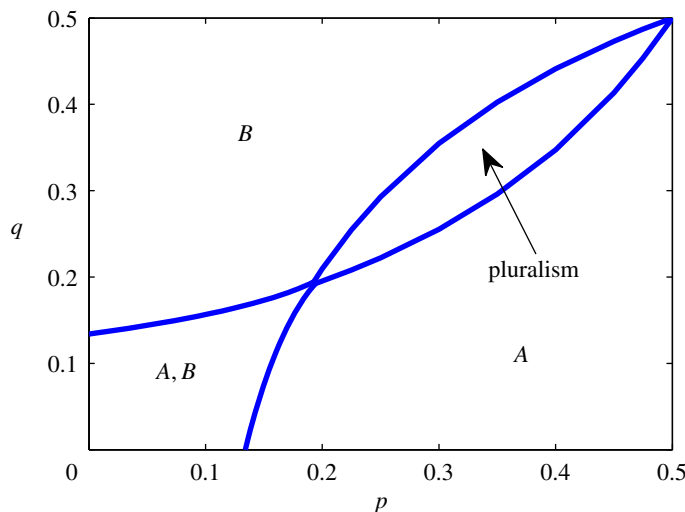
### (c) Pluralism and consensus

The dynamical picture through the stability diagram is complete for our purposes. We now use these dynamical results to explore the conditions which invoke *pluralism* in the system. Numerical simulations indicate there are no closed orbits for the ODE system, no matter the values of  $p$  and  $q$ . Consequently, in our definition we only need consider the stable fixed points, as initial data will generically lead to the subsequent solution converging to one of them. Mathematically, in this paper we define pluralism as follows.

**Definition 2.5.** Pluralism is present within the system when a stable fixed point satisfies  $0 < a + p, b + q \leq 0.5$ . We say that *consensus* is present within the system when all of the stable fixed points satisfy either  $a + p > 0.5$ , or  $b + q > 0.5$ .

**Remark 2.6.** It may be more accurate to say that our definition of consensus really only means majority. As there is no precise definition of consensus, we work with our current definition. It is important to point out here that changing the threshold for consensus, e.g. increasing it to 0.6, may lead to qualitatively different conclusions than those presented herein. In general, increasing the threshold will cause the set of values of  $p$  and  $q$  for which there is pluralism to grow, e.g. the ‘bubble’ in figure 3 will get larger. In particular, for  $p = q$  (whose bifurcation diagram is given in figure 2), if the consensus threshold is increased to 0.6, then the critical proportion decreases to  $p_c = (61 - 4\sqrt{115})/95 \sim 0.1906$ .

By studying the dynamics of the system—namely, through the fixed point analysis—we are able to determine the condition on the proportion of zealots which allows for pluralism to exist with respect to our opinion model. First suppose  $q = 0$ . The fixed point analysis shows that there is only one stable fixed point for  $p > p_b$  given by  $(a, b) = (1 - p, 0)$ . As  $a + p = 1$ , there will always be consensus associated with this fixed point; indeed, the entire population will necessarily have opinion A. On the other hand, using the results of proposition 2.2 there will also be consensus associated with the other fixed stable fixed point, as  $b = 1 - 2p - a > \frac{1}{2}$  for  $p \leq p_b$ . In conclusion, there is always consensus, and the value of  $p$  simply determines if either opinion can



**Figure 3.** The consensus/pluralism diagram for the one-city system (2.2) when  $0 \leq p + q \leq 1$ . There will be consensus for  $p, q \geq 0.5$ , so that part of the diagram is not shown. The horizontal axis is the proportion of zealots who hold opinion A, and the vertical axis is the proportion of zealots who hold opinion B. There is a bubble of pluralism which is symmetric with respect to the line  $q = p$  for  $0.1910 \leq p \leq 0.5$ ; otherwise, there is consensus. (Online version in colour.)

become dominant, or only one opinion is dominant. A summary cartoon for this analysis is given in figure 1.

Now suppose  $p = q$ . The fixed points are provided as in propositions 2.3 and 2.4. If  $p > \frac{1}{5}$ , there is only one stable fixed point, and it is straightforward to check that this fixed point is associated with pluralism. Consider the fixed points for  $p < \frac{1}{5}$ . The zeros for the nonlinear system (2.3) are invariant under  $(a, b) \mapsto (b, a)$ , so it is sufficient to determine those values of  $p$  for which

$$p + \frac{1}{2} \left( 1 - 3p + \sqrt{(1-p)(1-5p)} \right) > \frac{1}{2}.$$

Using the expression for the fixed point, and routine algebra applied to this inequality, provides consensus for

$$0 \leq p < p_c = \frac{1}{4} (3 - \sqrt{5}) \sim 0.1910;$$

otherwise, there is pluralism.

A summary cartoon for the analysis is given in figure 2. The  $a$ -value of the stable fixed point(s) for each value of  $0 \leq p \leq \frac{1}{2}$  are the solid curves. For  $p < \frac{1}{5}$ , the  $b$ -value of the fixed point is given by  $b = p^2/a$ . There is consensus for  $p < p_c$ , and pluralism otherwise. The opinion on which there is consensus depends upon the initial condition. In all cases, the number of moderates,  $m = 1 - a - b - 2p$ , is bounded above by  $\frac{1}{3}$ .

We finally consider the full picture where the number of zealots satisfies the constraints  $0 \leq p + q \leq 1$ . There is no analytic formula for the fixed points, so we rely upon the Matlab toolbox MATCONT to numerically generate the fixed points (see [30]). The results are presented in figure 3 (also see [31, fig. 1] for the related stability diagram which divides the number of possible fixed points for the system). There is always consensus if either of  $p$  or  $q$  is greater than 0.5. For  $0 \leq p, q \leq 0.5$ , we see there is a bubble of pluralism; otherwise, there is consensus. The bubble is symmetric with respect to the line  $p = q$ , which is a reflection of the symmetry  $(a, b, p, q) \mapsto (b, a, q, p)$ . The lower limit of the bubble,  $p = q = p_c$ , follows from the analysis associated with the case when the proportion of zealots is equal. As for the opinion on which there will be consensus, in the lower right it will always be opinion A, in the upper left it will always be opinion B and in the lower left the answer depends on the initial condition.



### 3. The $n$ -city ODE model

Recall that a city is a group for which the three-state mean-field model of opinion formation is operative. We now wish to extend our model to a collection of  $n$  cities, each of which in the absence of external influence is governed by the three-state mean-field model (2.2). We imagine that each city is a node on a graph (for an example of three cities, see figure 4). For each  $k = 1, \dots, n$  we apply the subscript  $k$  to each of the variables in (2.2). For the interactions between the cities, we have in mind the equivalent of table 1 (table 2). Whereas the mean-field model is derived as a limit of individual interactions, the model for multiple cities assumes that *groups* in one city are trying to influence groups in other cities.

Let  $i_{j,k} \geq 0$  denote the influence between the groups in city  $j$  with the groups in city  $k$ , and set  $I = (i_{j,k}) \in \mathcal{M}_n(\mathbb{R})$  to be the *influence matrix*. The influence matrix forms an opinion formation network, and through the influence matrix  $I$  we have a connected graph with weighted and directed edges. The influence matrix need not be symmetric. A cartoon example is provided in figure 4, which has the associated influence matrix,

$$I = \begin{pmatrix} i_{1,1} & i_{1,2} & 0 \\ i_{2,1} & 0 & i_{2,3} \\ 0 & i_{3,2} & i_{3,3} \end{pmatrix}.$$

We normalize the influence matrix,

$$\sum_{j=1}^n i_{j,k} \leq 1, \quad k = 1, 2, \dots, n. \quad (3.1)$$

In other words, the column sum for each column in the influence matrix is bounded above by one. In particular, each individual entry is also bounded above by one. If the column sum were one for each column, then the influence matrix would not only be an adjacency matrix for the graph, but it would also be a left stochastic matrix.

The appropriate modification to the original equations leads to the system for  $k = 1, 2, \dots, n$ ,

$$\begin{pmatrix} \dot{a}_k \\ \dot{b}_k \end{pmatrix} = M_k \begin{pmatrix} \sum_{j=1}^n i_{jk}(a_j + p_j) \\ \sum_{j=1}^n i_{jk}(b_j + q_j) \end{pmatrix}, \quad M_k = \begin{pmatrix} m_k & -a_k \\ -b_k & m_k \end{pmatrix}. \quad (3.2)$$

The influence matrix normalization of (3.1) guarantees that if the initial conditions satisfy the physical bounds for  $k = 1, 2, \dots, n$ ,

$$0 \leq a_k + p_k \leq 1, \quad 0 \leq b_k + q_k \leq 1 \quad \text{and} \quad 0 \leq m_k \leq 1, \quad (3.3)$$

then these bounds will be satisfied for all  $t \geq 0$ .

The system (3.2) has a more compact formulation. Set the individual opinion and zealot vectors,

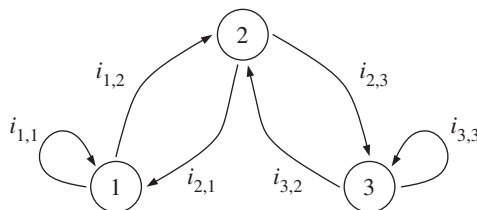
$$o_k = \begin{pmatrix} a_k \\ b_k \end{pmatrix} \quad \text{and} \quad z_k = \begin{pmatrix} p_k \\ q_k \end{pmatrix}$$

and let the total opinion and committed vectors be the concatenation,

$$O = \begin{pmatrix} o_1 \\ o_2 \\ \vdots \\ o_n \end{pmatrix} \in \mathbb{R}^{2n} \quad \text{and} \quad Z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{R}^{2n}.$$

Upon setting  $M$  to be the block-diagonal matrix,

$$M = \text{diag}(M_1, M_2, \dots, M_n) \in \mathcal{M}_{2n}(\mathbb{R}),$$



**Figure 4.** A cartoon of an opinion formation network with three cities. The variable  $i_{j,k}$  represents the individual influence of node  $j$  on node  $k$ . The total influence on a node, which is the sum of all the individual influences on that node, is normalized to be less than or equal to one.

**Table 2.** Interaction rule between city  $j$  and city  $k$ .

speaking group	listening group pre-interaction	listening group post-interaction
$A_j, P_j$	$B_k$	$(AB)_k$
	$(AB)_k$	$A_k$
$B_j, Q_j$	$A_k$	$(AB)_k$
	$(AB)_k$	$B_k$

the system (3.2) becomes

$$\dot{O} = M(I^T \otimes I_2)(O + Z), \quad (3.4)$$

where  $I_2 \in \mathcal{M}_2(\mathbb{R})$  is the identity matrix.

## 4. Case study: two cities

To begin to understand how the interaction between cities affects consensus and pluralism within each individual city, we start by analysing the case of two cities. We will simplify the analysis by assuming the column sum for each column of the influence matrix is one. A cartoon of the two-city network is given in figure 5. Here,  $i_1$  represents the influence that city 1 has on city 2, and  $i_2$  is the influence that city 2 has on city 1. The equations of motion (3.4) in this special case are

$$\left. \begin{aligned} \dot{o}_1 &= M_1[(1 - i_2)(o_1 + c_1) + i_2(o_2 + c_2)] \\ \dot{o}_2 &= M_2[i_1(o_1 + c_1) + (1 - i_1)(o_2 + c_2)]. \end{aligned} \right\} \quad (4.1)$$

We will investigate the effects of both the zealots and the influence factors on consensus and pluralism. We will consider several special cases, and unless said otherwise the analysis will be simplified by assuming  $i_1 = i_2 = i$ .

### (a) Zero zealots with equal influence

We begin by examining the special case where neither city contains zealots,  $c_1 = c_2 = 0$ . With  $i = i_1 = i_2$  the system (4.1) collapses to

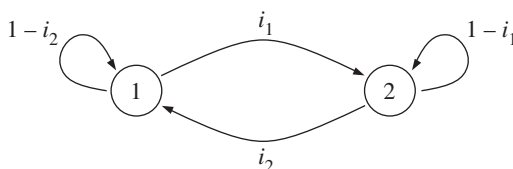
$$\dot{o}_1 = M_1[(1 - i)o_1 + io_2]$$

and

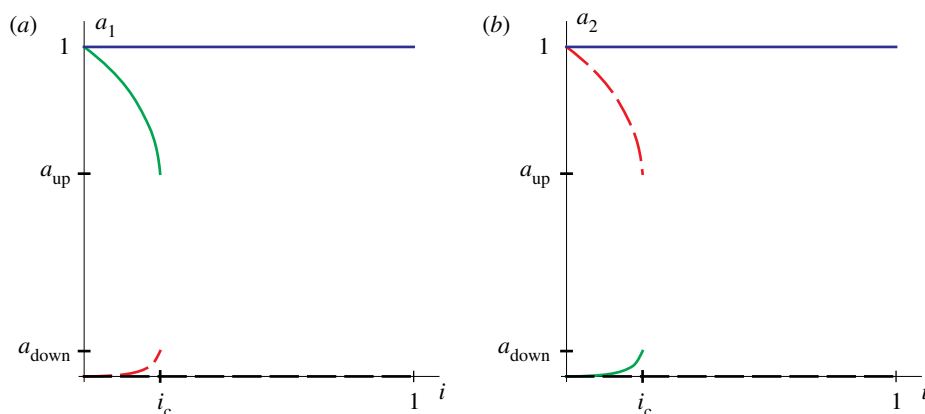
$$\dot{o}_2 = M_2[io_1 + (1 - i)o_2].$$

These equations are invariant under the mappings,

$$(o_1, o_2) \mapsto (o_2, o_1) \quad \text{and} \quad (a_1, b_1, a_2, b_2) \mapsto (b_1, a_1, b_2, a_2).$$



**Figure 5.** A cartoon of an opinion formation network with two cities.



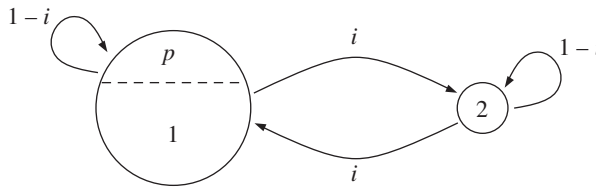
**Figure 6.** A cartoon of the stability diagram for the case of two cities, each of which has zero zealots. Only stable fixed points are represented. The horizontal axis represents the changing influence factor  $i = i_1 = i_2$ , and the vertical axis represents the proportion of the population holding opinion A within city 1 ( $a$ ) and city 2 ( $b$ ). The heavy solid (blue) horizontal lines correspond to the solution in which both populations share the same opinion A, and the heavy dashed (black) horizontal lines correspond to the solution in which both populations share the same opinion B. The thin solid (green) curve is the solution in which A is the majority opinion in city 1, and B is the majority opinion in city 2, whereas the thin dashed (red) curve is the opposite situation. This differing of opinion between the two cities is possible only for  $i < i_c \sim 0.167$ . The values on the vertical axis are  $a_{\text{up}} \sim 0.746$  and  $a_{\text{down}} \sim 0.054$ . (Online version in colour.)

Consequently, when plotting a bifurcation diagram as a function of  $i$ , it will be sufficient to plot only two of the four variables,  $a_1$  and  $a_2$ .

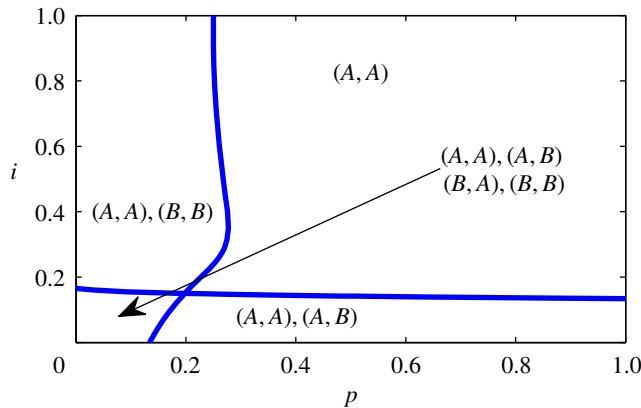
One set of solutions is  $\mathbf{o}_1 = \mathbf{o}_2 \in \{\mathbf{e}_1, \mathbf{e}_2\}$ , where  $\mathbf{e}_j \in \mathbb{R}^2$  is the standard unit vector. In other words, everyone in each city holds the same opinion. These fixed points are stable for any value of  $i$ . Regarding other possible solutions, a bifurcation diagram can be generated using AUTO [32] or MATCONT [30]. A cartoon of the final results is given in figure 6. The horizontal axis is the influence factor  $i$ , and the vertical axis is  $a_1$  and  $a_2$ . The curves represent stable fixed points (the unstable fixed points are not plotted). The solid/dashed thick lines (blue/black) correspond to the stable solution where both cities share the same opinion (solid is A, and dashed is B). The solid/dashed thin curves (green/red) correspond to the solution where one city has a majority who support opinion A, whereas the other has a majority who support opinion B. The transition values are numerically determined,

$$i_c \sim 0.167, \quad a_{\text{up}} \sim 0.746 \quad \text{and} \quad a_{\text{down}} \sim 0.054.$$

We see from the bifurcation diagram that both cities are always at a consensus. The critical influence value,  $i_c \sim 0.1667$ , in which the stability diagram goes from having four stable fixed points to two stable fixed points, plays the following role. For  $i > i_c$ , there is strong consensus: both cities have opinion A, or both cities have opinion B. However, for  $i < i_c$  it is possible to have consensus in which one city supports opinion A and the other city supports opinion B.



**Figure 7.** A cartoon of an opinion formation network with two cities when there are zealots in city 1,  $p_1 = p$ , and no zealots in city 2.



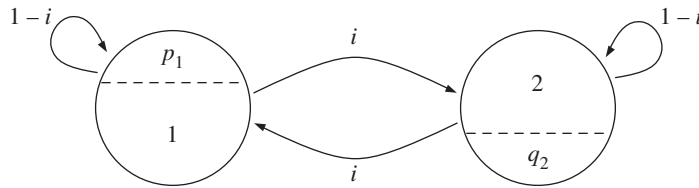
**Figure 8.** The consensus diagram for the case of zealots only in city 1. There is always consensus. The opinion on which consensus can occur in each city is labelled in the various regions. To the right of the vertical curve there is consensus in city 1 only on opinion A. Left of the vertical curve the situation is qualitatively the same as that for the case of no zealots: there will be consensus, but the opinion being A or B depends upon the initial data, and the level of inter-city influence. (Online version in colour.)

### (b) Committed believers in one city with equal influence

We now assume there are zealots in city 1 with  $p_1 = p$  and  $q_1 = 0$ , and no zealots in city 2,  $p_2 = q_2 = 0$ . A cartoon of the network is presented in figure 7. In the absence of inter-city influence, we know from our one-city analysis that if  $p > p_b$  city 1 will have full consensus on opinion A; otherwise, there will be consensus on either opinion. As for city 2, there will be consensus on either opinion.

To understand the situation for  $i > 0$ , we find the stable fixed points by using MATCONT to numerically continue from the known  $i = 0$  solutions. In doing so we find two curves on which saddle-node bifurcations occur. These are marked with a thick solid (blue) curve in figure 8. Although the coupled system is always at consensus (like the case of no zealots), we find that these curves separate  $(p, i)$ -space into four distinct regions regarding the type of consensus that is possible.

First, we note that a solution which exists for all  $i$  is  $(1 - p, 0, 1, 0)$ , which corresponds to both cities being in full agreement on opinion A. This is the only stable solution in the large upper right region in figure 8, i.e. the region in which both  $p$  and  $i$  are sufficiently large. To the right of the vertical curve, i.e.  $p > 0$  is sufficiently large, city 1 will always have consensus on opinion A; however, below the horizontal curve it is possible for city 2 to arrive at either opinion. There are more possibilities if the proportion of zealots is not too large (left of the vertical curve). If the inter-city influence is sufficiently large, then both cities will arrive at consensus on the same opinion. On the other hand, for small inter-city influence there are four different types of consensus which are possible. The one that is achieved depends upon the initial conditions.



**Figure 9.** A cartoon of an opinion formation network with two cities when each city has an equal number of zealots of the opposite persuasion,  $p_1 = q_2$ .

### (c) Symmetric zealots with equal influence

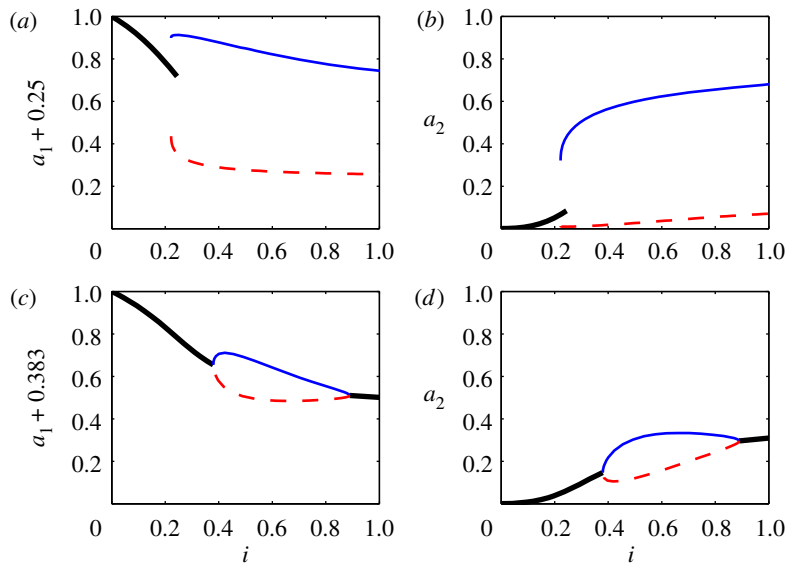
We now assume that each city has an equal proportion of zealots of the opposite persuasion,  $p_1 = q_2 = p$  with  $q_1 = p_2 = 0$  (figure 9). When there is no influence,  $i = 0$ , we know from §2c that each city will achieve consensus. The consensus must be of one opinion for a sufficiently large proportion of zealots; otherwise, it can be of either opinion. Moreover, if in city 1  $p_1 > p_b \sim 0.1340$ , then the consensus there will be opinion *A*, and if  $q_2 > p_b$ , the consensus in city 2 will be opinion *B*. Finally, we saw in §4a that in the absence of zealots consensus will be achieved in both cities, and if  $i$  is sufficiently large, the consensus in both cities will be of the same opinion.

To determine the region in  $(p, i)$ -space for which there could be consensus for  $i > 0$  we first need to find all of the stable fixed points. When  $p \geq 0.5$ , there is always consensus (*A* in city 1 and *B* in city 2); thus, we only need consider  $p < 0.5$ . The fixed points are found by using MATCONT and continuing from the known  $i = 0$  solution(s). For  $p_1 = q_1 \leq p_b$  the bifurcation diagram is qualitatively similar to that of figure 6 for the problem of no zealots: there will always be consensus, and the form of the consensus depends upon the initial conditions. Consequently, we look to better understand what happens when  $p > p_b$ , i.e. when in the case of zero inter-city influence each city is at consensus.

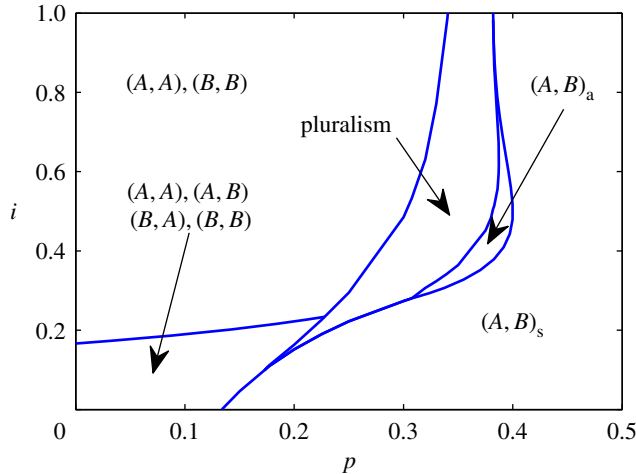
When  $i = 0$ , there is a unique solution,  $(a_1, b_1, a_2, b_2) = (1 - p, 0, 0, 1 - p)$ . If  $p \geq 0.4$ , we find the continuation of this solution to be the only stable solution; moreover, it always corresponds to consensus. The situation becomes more interesting for  $p_b \leq p \leq 0.4$ . First, when  $p = 0.4$ , there is a bifurcation of the continued fixed point when  $i = 0.5$ . A ‘bubble’ of stability appears, and each end of the bubble connects to the main branch via a pitchfork bifurcation. A sample plot for  $p = 0.383$  of such a bifurcation is given in figure 10*c,d*. Figure 10*a,c* corresponds to the total proportion of those who hold opinion *A* in city 1,  $a_1 + 0.383$ , and figure 10*b,d* gives the proportion of those who hold opinion *A* in city 2. The thick solid (black) curves correspond to the solution which is the continuation of the  $i = 0$  solution. The thin solid (blue) curves are in correspondence, as are the thin dashed (red) curves.

As  $p$  continues to decrease, the right-most bifurcation point occurs for  $i > 1.0$ , so it is no longer physically relevant. However, for the left-most point the pitchfork bifurcation is still supercritical for  $p > 0.3075$ . Further decreasing the value of  $p$  reveals that the supercritical bifurcation becomes subcritical at  $p \sim 0.3075$ , and for smaller values of  $p$  there is hysteresis. A sample plot for  $p = 0.25$  of such a bifurcation is given in figure 10*a,b*. Figure 10*a,c* corresponds to the total proportion of those who hold opinion *A* in city 1,  $a_1 + 0.25$ , and figure 10*b,d* gives the proportion of those who hold opinion *A* in city 2. Again, the thick solid (black) curves correspond to the solution which is the continuation of the  $i = 0$  solution, the thin solid (blue) curves are in correspondence, and so are the thin dashed (red) curves.

The stability diagrams become even more complicated as  $p \rightarrow 0^+$ , so we now focus on the consensus/pluralism diagram, which is given in figure 11. Here, we see that there is consensus in *both* cities except in the region marked ‘pluralism’. In that region, one city will be at consensus (*A* in city 1, and *B* in city 2), but the other will not. The horizontal bounds for which it is possible to have pluralism in at least one city is approximately  $0.1726 \leq p \leq 0.3872$ . Regarding the regions where the consensus is on *A* in city 1 and *B* in city 2, the subscript ‘s’ means that the proportion of those who hold opinion *A* in city 1 is precisely that who hold opinion *B* in city 2, and the subscript



**Figure 10.** The stability plots for  $p = 0.25$  (*a,b*), and  $p = 0.383$  (*c,d*). Only stable solutions are plotted. The plots in (*a,c*) are for  $a_1 + p$  versus  $i$ , and those in (*b,d*) are for  $a_2$  versus  $i$ . In (*a,c*), we plot the total proportion of those who believe opinion *A* in city 1, and in (*b,d*) we plot the total proportion of those who believe opinion *A* in city 2. The thick solid (black) curve corresponds to the continuation of the  $i = 0$  solution where each city is at full consensus with differing opinion. The thin solid (blue) curves correspond to each other, as do the dashed (red) curves. (Online version in colour.)



**Figure 11.** The bifurcation and pluralism plot for the case  $p_1 = q_2 = p$ . There is consensus in both cities except in the region marked 'pluralism'. In that region it may be possible there is consensus in one city, but not the other. The subscript 's' means that the proportion of those holding opinion *A* in city 1 is the same as those holding opinion *B* in city 2. The subscript 'a' means that, while there is still consensus in each city, this particular symmetry is broken. (Online version in colour.)

'a' means this symmetry has been broken (through the bifurcation associated with figure 10*c,d*). In conclusion, unlike the previous cases the presence of zealots and inter-city influence allows for a region where one of the cities will not reach consensus. Outside of that region, either the inter-city influence is the dominant effect (left of the pluralism region), or the presence of zealots is the dominant effect (right of the pluralism region).

## 5. Dynamics with many cities

We finally consider the problem of having many interacting cities. The goal here will be to use our understanding of the stability problem, which was used to generate the previous consensus/pluralism diagrams, to begin to understand the process of clustering of opinion. By clustering we mean a group of adjacent cities (where adjacency is defined by the directed graph associated with the influence matrix) sharing the same opinion. In our analysis, we will only consider the scenario where there are no zealots in any of the interacting cities. Our (numerically assisted) rigorous analysis in §4a indicates that for two cities there will always be consensus, and the type of consensus depends on the amount of interaction between the two cities. In particular, for weak inter-city interaction any combination is possible, but if the inter-city interaction is sufficiently large, the two cities will agree on the consensus opinion. Our expectation is that there will be a similar situation for many cities. However, we further expect that these thresholds where neighbouring cities must agree will depend on the amount of clustering already present in the system. We will not do a full study; instead, we will simply focus on a couple of representative graphs.

### (a) Case study: cycle graph

Initially place the individual cities on a line. For our case study we assume nearest-neighbour interaction only, i.e. city  $j$  directly interacts with cities  $j - 1$  and  $j + 1$  only. Moreover, we assume a cycle graph, so if there are  $n$  cities, then city  $n$  interacts with city 1, and city 1 also interacts with city  $n$  (see figure 12 for the case of  $n = 8$ ). The influence matrix is symmetric with  $I_{j,j} = 1 - \epsilon$ ,  $I_{j-1,j} = I_{j,j-1} = \epsilon/2$  and  $I_{1,n} = I_{n,1} = \epsilon/2$ . At this point in time the problem is beyond a rigorous (or numerically assisted) analysis; instead, we do some numerics to gain insight into the dynamics.

Regarding the steady-state solutions, when  $\epsilon = 0$ , the cities are all uncoupled. Consequently, the steady-state behaviour is precisely that associated with the one-city model studied in §2. For each city the only steady states are

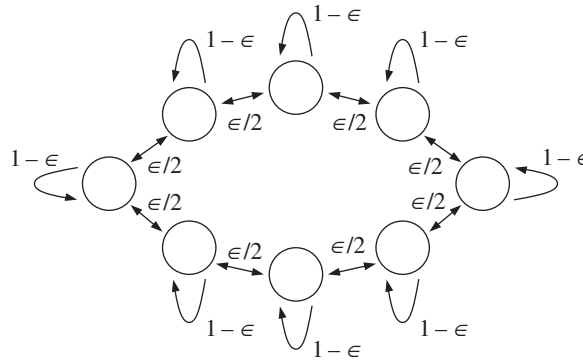
$$(a_j, b_j) \in \{(1, 0), (0, 1), (\frac{1}{3}, \frac{1}{3})\}.$$

For the purposes of labelling, set

$$O_a = (1, 0), \quad O_b = (0, 1) \quad \text{and} \quad O_{1/3} = (\frac{1}{3}, \frac{1}{3}).$$

For the one-city model both  $O_a$  and  $O_b$  are stable, and the linearization has two real and negative eigenvalues, while  $O_{1/3}$  is an unstable saddle point. A steady state for the full system when  $\epsilon = 0$  can be thought of as a vector where each entry is one of  $O_a, O_b$  or  $O_{1/3}$ . This steady state will be stable for the full system if none of the entries is  $O_{1/3}$ , and all of the eigenvalues of the linearization will be real and negative; otherwise, each entry which has  $O_{1/3}$  will bring with it one real and positive eigenvalue. A steady state for the full system will be hyperbolic, so by the implicit function theorem (IFT) there will be an  $\epsilon_0 > 0$  such that this solution will continue and have the same type of stability for  $\epsilon < \epsilon_0$ . Consequently, for small inter-city interactions the results are derived from those of the one-city model. In particular, if the original steady state has no entries with  $O_{1/3}$ , then the continued state will be stable.

Assume the state when  $\epsilon = 0$  has no entries with  $O_{1/3}$ , so it is stable. As  $\epsilon$  increases, the IFT can no longer be relied upon to provide stable solutions as continuations from the uncoupled case. For an example of what can happen, assume there are  $n = 50$  cities. Assume that when  $\epsilon = 0$ , cities 1 through  $1 + m$  are  $O_a$ , while the remaining cities are  $O_b$  ( $1 + m$  adjoining cities share opinion  $A$ , while the rest share opinion  $B$ ). Using the Matlab function `fsolve`, we numerically continue this solution. We find, for  $0 \leq m \leq 3$ , there is a saddle-node bifurcation at  $\epsilon = \epsilon_{SN}$  (see table 3); however, for  $m \geq 4$  there is no such bifurcation, and the solution continues and is stable until  $\epsilon \sim 0.99$ . For the case of two cities discussed in §4a the saddle-node bifurcation takes place at a slightly larger value,  $\epsilon_{SN} \sim 0.1667$ . Note the generic (and probably expected) phenomenon that the more cities



**Figure 12.** The cycle graph network for  $n = 8$  cities in which there is nearest-neighbour interaction.

**Table 3.** The values for which a saddle-node bifurcation takes place for 50 cities under the assumption that, when  $\epsilon = 0$ , city 1 through city  $1 + m$  have opinion  $A$ , and all the other cities have opinion  $B$ .

$m$	0	1	2	3	$\geq 4$
$\epsilon_{\text{SN}}$	0.1410	0.3134	0.4879	0.6291	$> 0.99$

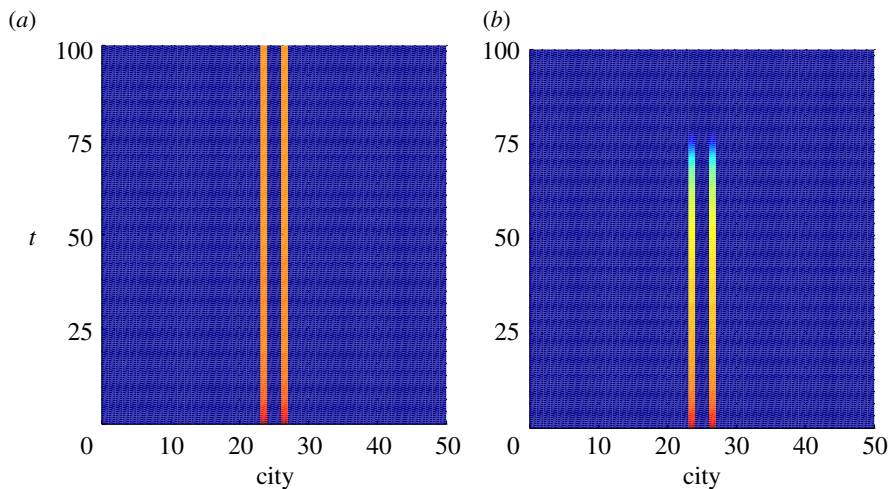
who hold the same opinion and directly talk to each other, then the more outside influence this smaller subnetwork can withstand without changing its opinion.

What happens dynamically to perturbations of a steady state near the saddle-node bifurcation point? For the sake of clarity, suppose  $m = 1$ , and for  $\epsilon < 0.3134$  write the solution at  $\epsilon = 0$  as  $(O_a, O_a, O_b, \dots, O_b)$ . It is seen numerically that an initial condition of  $(1, 1, 0, \dots, 0)$  (with each entry having a bit of random noise added) decays to the stable solution. On the other hand, suppose  $\epsilon > 0.3134$  (but not by too much). It is seen that this same initial condition initially appears to converge to the seeming continuation of  $(O_a, O_a, O_b, \dots, O_b)$  (which no longer exists) before undertaking a quick transition and eventually converging to the solution  $(O_b, O_b, O_b, \dots, O_b)$  (all cities share opinion  $B$ ). Now, the initial condition could also be thought of as a perturbation of the continuation of the unstable solution,  $(O_a, O_a, O_{1/3}, O_b, \dots, O_b, O_{1/3})$ , which continues to exist for  $\epsilon > 0.3134$ . When  $n = 50$ , this solution has an unstable manifold of dimension two, and a stable manifold of dimension 98. Consequently, the trajectory attached to the initial condition initially appears to converge to the steady state  $(O_a, O_a, O_{1/3}, O_b, \dots, O_b, O_{1/3})$  by following the stable manifold before eventually following the unstable manifold to its final destination.

We now consider the effect of having several clumps of cities, with each clump having a different shared opinion. The primary idea behind the following discussion is an idea familiar to those who study the dynamics of interacting waves: if the waves are sufficiently separated, then to leading order the dynamical behaviour of each individual wave ignores the adjacent waves (e.g. see [33–36] and [37, ch. 10.7]). Thus, we can use the results of table 3 to better understand the dying and merging of opinion as a function of time.

For our first example, take two cities having opinion  $A$ , and separate them by two cities which share opinion  $B$ ,  $(\dots, O_b, O_a, O_b, O_b, O_a, O_b, \dots)$ . Suppose  $\epsilon < 0.1410$ . As each individual city is itself stable, the expectation is that this configuration will be stable, and the two cities will retain opinion  $A$ . This is precisely what is seen in figure 13a. Now suppose  $\epsilon > 0.1410$ , so the continuation of the stable solution  $(O_a, O_b, \dots, O_b)$  does not exist, but the continuation of the unstable solution,  $(O_a, O_{1/3}, O_b, \dots, O_b, O_{1/3})$ , does exist. For one city the initial condition  $(O_a, O_b, \dots, O_b)$  can be thought of as a perturbation of the unstable solution. The dynamics are such that for some time the solution will look like the continuation of (the non-existent)  $(O_a, O_b, \dots, O_b)$ , but then there will be a quick transition to the stable solution,  $(O_b, O_b, \dots, O_b)$ . The expectation is that the configuration





**Figure 13.** Two simulations when cities 24 and 27 initially hold opinion *A* (red), and all other cities hold opinion *B* (blue). A vertical strip in each panel represents the time evolution of opinion in a particular city. The saddle-node bifurcation point for an individual city is  $\epsilon_{SN} \sim 0.1410$ . (a) The result when  $\epsilon = 0.137$ , and (b) the result when  $\epsilon = 0.145$ . In (a), cities 24 and 27 retain opinion *A* for all time, while in (b) each of these cities changes over to opinion *B* after approximately 75 time units. (Online version in colour.)

with two cities will have the same behaviour: first, it will appear as if each city retains opinion *A*, but then there will be a quick transition to both cities having opinion *B*. This is precisely what we see from the simulation results presented in figure 13*b*.

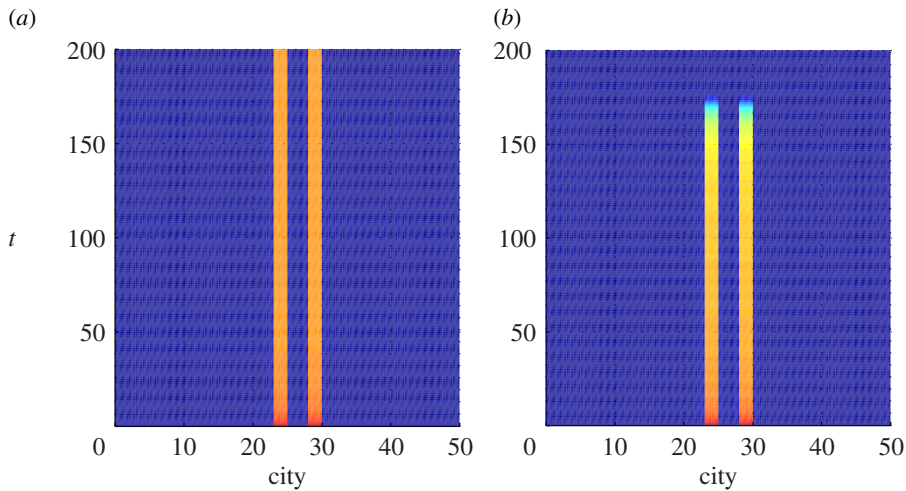
For our second example, suppose we take two groups of two cities having opinion *A*, and separate the two groups by a group of three cities having opinion *B*. The expectation is that, for  $\epsilon < 0.3134$ , this configuration will be stable, and each of the two cities will retain opinion *A*. This is what is seen in figure 14*a*. On the other hand, if  $\epsilon > 0.3134$ , then, as we have already seen, a single group of two cities will eventually change their opinion and take on the surrounding opinion. Consequently, we expect the same for two groups of two cities, and this is what is seen in figure 14*b*.

For our final example, we take two groups of three cities having opinion *A*, and separate the two groups by a group of two cities having opinion *B*. The expectation is that, for  $\epsilon < 0.3134$ , this configuration will be stable, and each of the two outside groups will retain opinion *A*, and the inside group will retain opinion *B*. This is what is seen in figure 15*a*. On the other hand, if  $\epsilon > 0.3134$  (but not by too much), then the group of two cities no longer exists as a steady-state solution, and dynamically we expect that a perturbation will eventually take on the opinion of the surrounding groups. Consequently, we expect that eventually there will be a group of eight cities which share opinion *A*, which will be stable. This is exactly what we see in figure 15*b*.

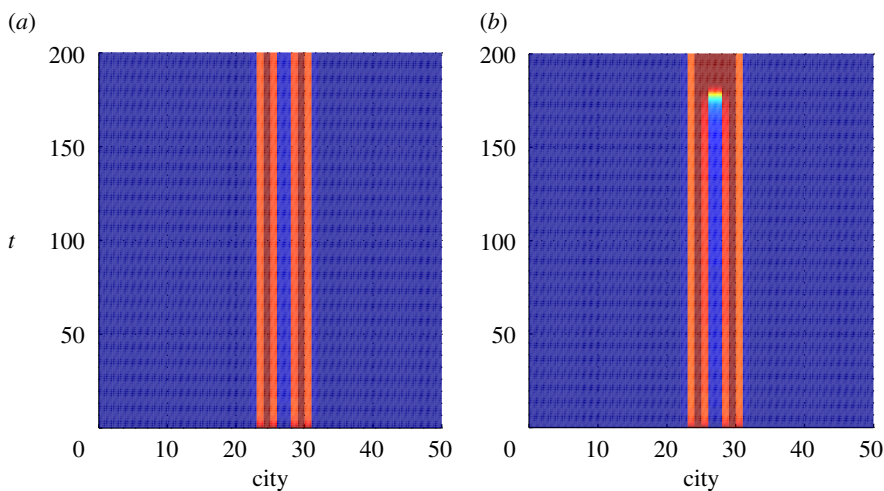
In conclusion, we see strong evidence for the conjecture that the dynamics associated with widely separated clusters are essentially determined by the dynamics associated with a single cluster. In particular, the saddle-node bifurcation point associated with the cluster size provides a threshold for a cluster of one opinion to form the opinion of its neighbours.

## (b) Case study: cycle graph with a hub

We finally, and briefly, consider the case of a cycle graph with the addition of a hub at the centre (figure 16). The hub city has influence  $i$  on each of the cities in the cycle, whereas each city in the cycle has influence  $i/n$  on the hub city. The cities on the cycle interact with each other in the same nearest-neighbour manner as described in §5*a*. We consider the question of what happens



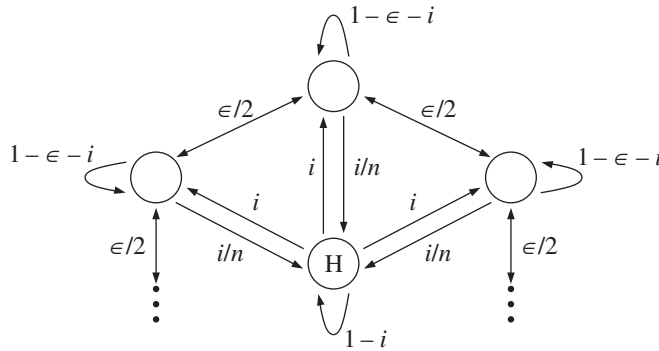
**Figure 14.** Two simulations when cities 24–25 and 29–30 initially hold opinion *A*, and all other cities hold opinion *B*. The saddle-node bifurcation point for an individual group is  $\epsilon_{\text{SN}} \sim 0.3134$ . (a) The results when  $\epsilon = 0.310$ , and (b) results when  $\epsilon = 0.316$ . (Online version in colour.)



**Figure 15.** Two simulations when cities 24–26 and 29–31 initially hold opinion *A*, and all other cities hold opinion *B*. The saddle-node bifurcation point for the individual group of two is  $\epsilon_{\text{SN}} \sim 0.3134$ . (a) The results when  $\epsilon = 0.310$ , and (b) the results when  $\epsilon = 0.316$ . (Online version in colour.)

to overall opinion on the network if there are zealots in the hub. In the special case that each of the cities on the cycle graph has the same opinion, the system collapses to the two-city model discussed in §4. In the cartoon of figure 5, the hub city is city 1, the conglomeration of cities on the cycle graph is city 2 and the inter-city reaction rates satisfy  $i_1 = i_2 = i$ . Note that in this submanifold of solutions we have consensus/pluralism diagrams exactly as presented therein.

For a particular example, consider the situation presented in §4b. The hub has a zealot subpopulation who hold opinion *A*, while the cities on the cycle graph have no zealots of either persuasion. As we see from figure 8, if the proportion of zealots is sufficiently large ( $p$  lies to the right of the vertical curve), then the hub will have consensus on opinion *A* no matter the opinion in the cities on the cycle. Moreover, if the interaction is sufficiently large ( $i$  is above the



**Figure 16.** A cycle graph with a hub. The cities on the cycle graph are not labelled, whereas the hub city is denoted by 'H'. Each city on the cycle graph is connected to a hub city. On the cycle graph, the  $n$  cities have nearest-neighbour interaction with strength  $\epsilon/2$ . The hub interacts with each city on the circle with strength  $i$ , while each city interacts with the hub with strength  $i/n$ .

horizontal curve), then each city on the cycle graph will also have consensus on opinion  $A$ . In other words, if  $(p, i)$  are chosen to be in the upper right portion of the figure, then *all* cities share opinion  $A$ . Note that this conclusion does not require a large proportion of zealots in the hub city, nor a large influence on the cycle cities from the hub city. While we have not done a formal analysis, numerical simulations indicate that this configuration is stable for the full system even if the opinions in each of the cities on the cycle graph are not assumed to be identical.

## 6. Conclusion

We reconsidered the three-state mean-field opinion model of [23] from the perspective of pluralism and consensus associated with the stable fixed points. This allowed for a simplification of the presentation of the results; in particular, it allowed for a better understanding of the role of zealots in opinion formation.

We extended this model to an ODE on a graph, where the nodes correspond to individual cities, and the edges correspond to reaction rates between the cities. In the case of two cities, we discuss how the (perhaps competing) presence of zealots in one or both of the cities affects consensus. In particular, we show that if one city has enough zealots, and the interaction between the two cities is sufficiently strong, then there will be consensus in both cities on the opinion shared by the zealots.

In the event that there are many cities, we considered two case studies: a cycle graph, and a cycle graph with a hub. In the case of the cycle graph, we considered the problem of opinion clustering, and demonstrated via some elementary stability analysis that the clustering phenomena could be understood (at least in some cases) as a consequence of a saddle-node bifurcation. In the case of the cycle graph with a hub, we looked at the situation where there are zealots in the hub, and considered how the opinion of the zealots could propagate and dominate the entire network.

There are many subsequent questions to be considered, especially when there are many cities. Examples include the following:

- What if the influence between cities was a function of similarity, i.e. what if the rate increased if cities shared the same opinion, but decreased if cities differed in opinion?
- How does the distribution of zealots across the cities affect opinion formation, and in particular the rise of a network-wide consensus? Is it better to influence the network by concentrating all the zealots in a small number of cities, or should the zealots be widely distributed?

- (c) To what extent can dominant voice models be generalized through this network structure? In particular, what happens to opinion formation on the whole network if there are competing hubs?

These, and others, will be the subject of future research.

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